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The impact of the real interest rate on the optimal rotation age in the Hartman model: A correction of comparative statics of real interest rate in Amacher et al. 2009 (October 2012)

This note contains a corrected proof of comparative static effect of real interest rate presented in Amacher et al. 2009, pp.57-58. We thank Dick Brazee bringing the error in this proof to our attention, and Olli Kuusela for critical suggestions. This note revisits the proof and the entire problem, and in doing so we have a generalized proof that corrects our error and shows the following: there is an inverse relationship between rotation age and the interest rate when young stands are valued for amenities stands (from Amacher et al. 2009, this means in the sense that *F'*(*s*) < 0). However, the effect is ambiguous when amenities from old stands are valued (stands in the sense that *F'*(*s*) > 0).

In the Hartman model the landowner values both harvest revenue from a planted stand of trees and amenity benefits that are jointly provided by this stand. Thus, the generic joint production Hartman model is defined as it is in his seminal paper,

$$W = \left[pf(T)e^{-rT} + \int_0^T F(s)e^{-rs}ds - c \right](1-e^{-rT})^{-1}. \tag{H}$$

The optimal rotation age in the Hartman model is determined by the following first-order condition:

$$pf(T) - rpf(T) - rV + F(T) - rE = 0, \tag{1}$$

where $V = (pf(T)e^{-rT} - c)(1-e^{-rT})^{-1}$ is the "Faustmann part"

and $E = \int_0^T F(s)e^{-rs}ds(1-e^{-rT})^{-1}$ is the "Hartman part".

The comparative static effect of the real interest rate is given by (see equation 3.7b in Amacher et al. 2009)

$$\frac{\partial T}{\partial r} = -\frac{W_{Tr}}{W_{TT}}, \tag{2}$$

where

$$W_{Tr} = -(V + r\frac{dV}{dr} + E + r\frac{dE}{dr}). \tag{3}$$

At the interior solution $W_{TT} < 0$, so that the sign of W_{Tr} determines the sign of equation (2). From the analysis in Chapter 2, we know the interest rate impact on the Faustmann part is unambiguously negative $-(V+r\frac{dV}{dr})<0$. Thus, to see how the real interest rate impacts in the Hartman model, we need to determine the sign of $W_{Tr}=-(E+r\frac{dE}{dr})$.

Notice first that,

$$(E+r\frac{dE}{dr})=E-\frac{r\int_0^T sF(s)e^{-rs}ds}{(1-e^{-rT})}-\frac{rTe^{-rT}}{(1-e^{-rT})^{-2}}\int_0^T F(s)e^{-rs}ds\ .$$

The second term contains term *E*, so that we can re-express the above equation as,

$$(E+r\frac{dE}{dr})=\left[1-\frac{rTe^{-rT}}{(1-e^{-rT})}\right]E-\frac{r\int_0^T sF(s)e^{-rs}ds}{(1-e^{-rT})} \tag{4}$$

We next evaluate the integral term: $\int_0^T sF(s)e^{-rs}ds$ in equation (4) by partial integration. To this end we define the terms as follows: $v=sF(s)$ and $u'=e^{-rs}$. Thus, we have:

$v=sF(s)$, so that $v'=F(s)+sF'(s)$
$u=-\frac{1}{r}e^{-rs}$, so that $u'=e^{-rs}$

Using these definitions we integrate by parts:

$$\int_0^T sF(s)e^{-rs}ds=-\Big|_0^T sF(s)\frac{e^{-rs}}{r}+\int_0^T(F(s)+sF'(s))\frac{e^{-rs}}{r}ds$$

Hence, we have get,

$$\int_0^T sF(s)e^{-rs}ds=-\frac{TF(T)e^{-rT}}{r}+\int_0^T(F(s)+sF'(s))\frac{e^{-rs}}{r}ds\ . \tag{5}$$

We next plug equation (5) in equation (4) to obtain,

$$(E+r\frac{dE}{dr})=\left[1-\frac{rTe^{-rT}}{(1-e^{-rT})}\right]E+\frac{rTF(T)e^{-rT}}{r(1-e^{-rT})}-\frac{r\int_0^T(F(s)+sF'(s))\frac{e^{-rs}}{r}ds}{r(1-e^{-rT})}$$

By eliminating *r* and decomposing the integral terms we obtain,

$$(E + r\frac{dE}{dr}) = \left[1 - \frac{rTe^{-rT}}{(1-e^{-rT})}\right]E + \frac{TF(T)e^{-rT}}{(1-e^{-rT})} - \frac{\int_0^T (F(s)e^{-rs}ds}{(1-e^{-rT})} - \frac{\int_0^T sF'(s)e^{-rs}ds}{(1-e^{-rT})}$$

Noting that the first integral term is equal to *E* by definition, we re-express this equation as,

$$(E + r\frac{dE}{dr}) = \left[1 - \frac{rTe^{-rT}}{(1-e^{-rT})}\right]E + \frac{Te^{-rT}F(T)}{(1-e^{-rT})} - E - \frac{\int_0^T sF'(s)e^{-rs}ds}{(1-e^{-rT})}$$

By subtracting, rearranging and accounting for the minus sign in equation (3), we end up with

$$-(E + r\frac{dE}{dr}) = -\left(\frac{Te^{-rT}}{(1-e^{-rT})}\left[F(T) - rE\right] - \int_0^T sF'(s)e^{-rs}ds(1-e^{-rT})^{-1}\right). \tag{6}$$

We can work with equation (6). Consider first the bracket term, $F(T) - rE$. Using the definition of *E*, we have,

$$F(T) - rE \ = \ F(T) - \frac{1}{(1-e^{-rT})}\int_0^T rF(s)e^{-rs}ds \ .$$

By partial integration we obtain

$$F(T) - rE = F(T) - \frac{1}{1-e^{-rT}}\left(-\Big|_0^T F(s)e^{-rs} + \int_0^T F'(s)e^{-rs}ds\right)$$

$$F(T) - rE = F(T) - \frac{1}{1-e^{-rT}}\left(-F(T)e^{-rT} + \int_0^T F'(s)e^{-rs}ds\right),$$

Which gives by collecting terms:

$$F(T) - rE = \frac{1}{1-e^{-rT}}\left(F(T) - \int_0^T F'(s)e^{-rs}ds\right) \tag{7}$$

By the definition of an integral, we can state that $F(T) = \int_0^T F'(s)ds$. We use this integral to replace *F*(*T*) in equation (7) and obtain,

$$F(T)-rE=\frac{1}{1-e^{-rT}}\left(\int_0^T F'(s)ds-\int_0^T F'(s)e^{-rs}ds\right)=\frac{1}{1-e^{-rT}}\left(\int_0^T F'(s)(1-e^{-rs})ds\right) \qquad (8)$$

We next substitute equation (8) to equation (6) and after straightforward manipulation we have:

$$-(E+r\frac{dE}{dr})=\int_0^T F'(s)\left(\frac{Te^{-rT}(1-e^{-rs})-se^{-rs}(1-e^{-rT})}{(1-e^{-rT})^2}\right)ds\,. \qquad (9)$$

We must still determine the sign of the term in parenthesis. The denominator is always positive. Define the numerator as $G(s)=Te^{-rT}(1-e^{-rs})-se^{-rs}(1-e^{-rT})$. Using $G(s)$ we can show that the term inside the parenthesis in equation (9) is always *negative*. This can be established by first noting that $G(s)=0$ when $s=0$; and $G(s)=0$ when $s=T$. Consequently, we know using the mean value theorem that $G(s)$ has a critical point $s^*\in(0,T)$. Our goal is to establish that s^* is unique and that $G(s^*)<0$.

To find the critical point, we differentiate $G(s)$ with respect to s and solve for s. From the derivative $e^{-rs}\left[rTe^{-rT}-(1-rs)(1-e^{-rT})\right]=0$, we note that the first term, e^{-rs}, is zero only asymptotically. Therefore we have,

$$s^*=\frac{1}{r}\left(1-\frac{rTe^{-rT}}{1-e^{-rT}}\right)$$

This establishes that s^* is unique.

To establish that s^* is a minimum and has a negative value we show that

$$\lim_{s\to 0}G'(s)=e^{-rT}(rT+1)-1<0$$

Therefore, at $G(0)=0$, the function is strictly decreasing and achieves its minimum value at point s^* and $G(s^*)<0$ holds. The fact that $e^{-x}(x+1)<1$ can be established by a simple graphical argument: the graph of e^x lies always above of $x+1$ and hence is greater than $x+1$ for all values $x>0$. This fixes the argument that numerator of (9) is negative.

We can now determine the sign of the Hartman term, $W_{Tr}=-(E+r\frac{dE}{dr})$:

$$W_{Tr}=-(E+r\frac{dE}{dr})>0 \;\; if \;\; F'(s)>0 \qquad (10a)$$

$$W_{Tr}=-(E+r\frac{dE}{dr})<0 \;\; if \;\; F'(s)<0 \qquad (10b)$$

Recall, the impact of the interest rate on the Faustmann part is unambiguously negative. Therefore, we have

$$\frac{dT^H}{dr} = ?\ if\ \ F'(s) > 0 \tag{11a}$$

$$\frac{dT^H}{dr} < 0\ if\ \ F'(s) < 0 \tag{11b}$$

$$\frac{dT^H}{dr} < 0\ if\ \ F'(s) = 0 \tag{11c}$$

Hence, we have shown the following result, QED:

The impact of an increase in the real interest rate on the optimal Hartman rotation age depends on the landowner's valuation of amenity benefits:

- If the landowner values amenities from old stands (in the sense that $F'(s) > 0$), then a higher interest rate tends to shorten the optimal rotation age via timber production impact but is counter-affected by the amenity valuation impact which tends to lengthen the optimal rotation age, so that the overall interest rate effect is ambiguous.
- If the landowner values amenities from young stands (in the sense that $F'(s) < 0$), then the optimal rotation age unambiguously shortens, because both timber production impact and amenity valuation impact reinforce each other.
- If the landowner values amenities from young stands (in the sense that $F'(s) = 0$), then the rotation age unambiguously shortens, because only timber production impact is operative and it unambiguously shortens the optimal rotation age.